Solutions

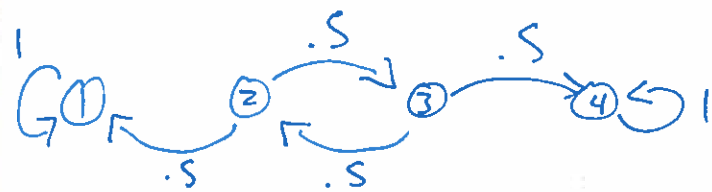
7.7: Markov Systems

Example 1. You go to a casino armed with your annual bonus of \$20. You have a very simple betting strategy. You will play roulette and on each spin of the wheel you will place \$10 on red. For simplicity, we will assume that the probability of winning is 1/2. If red comes up, you will win an additional \$10 and if black comes up you lose your \$10. You decide to keep playing until either you have up to \$30 or you lose it all. What is the probability that you walk away with \$30?

	Money	Probability
Time 0:	\$20	1
Time 1:	\$10	1/2
	\$30	1/2
Time 2:	\$0	1/4
	\$20	1/4
	\$30	1/2
Time 3:	\$0	1/4
	\$10	1/8
	\$30	5/8
Time 4:	\$0	5/16
	\$20	1/16
	\$30	11/16
Time 5:	\$0	5/16
	\$10	1/32
	\$30	23/32

.
.
.

1 = \$0, 2 = \$10, 3 = \$20, 4 = \$30



Transition Matrix: (from row to column)

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Initial Distribution: $v = (0\ 0\ 1\ 0)$

Time 1: $vP = (0\ \ 1/2\ \ 0\ \ 1/2)$

Time 2: $vP^2 = (1/4\ \ 0\ \ 1/4\ \ 1/2)$

Time 3: $vP^3 = (1/4\ \ 1/8\ \ 0\ \ 5/8)$

Time 4: $vP^4 = (5/16\ \ 0\ \ 1/16\ \ 11/16)$

~~Steady State:~~

Time 50: $vP^{50} \approx [.\overline{33}\ \ 0\ \ 0\ \ .\overline{66}]$

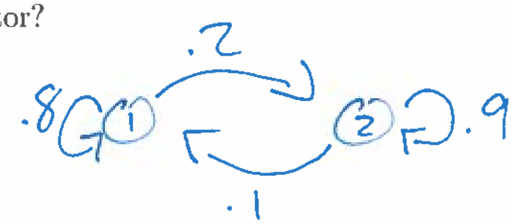
Example 2. Suppose you are running a study on whether consumers prefer liquid or powdered laundry detergent. You find that 20% of those who used powdered detergent at the beginning of the year switched to liquid by the end of the year; whereas, only 10% who used liquid detergent at the beginning of the year switched to powder by the end of the year. Suppose that at the begging of the year 70% used powder and the remaining 30% used liquid.

(a) What will the distribution look like one year from now?

(b) What will the distribution look like five years from now?

(c) What is the steady state distribution vector?

Powder = 1   Liquid = 2



$$P = \begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix} \qquad v = (.7 \ \ .3)$$

(a) $vP = (.7 \ .3)\begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix} = (.59 \ \ .41)$

(b) $vP^5 = (.7 \ .3)\begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix}^5 = (.7 \ .3)\begin{pmatrix} .445 & .554 \\ .277 & .723 \end{pmatrix} = (.395 \ \ .605)$

(c) Want $vP = v$.

$$(x \ y)\begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix} = (x \ y) \Rightarrow \begin{aligned} .8x + .1y &= x \\ .2x + .9y &= y \end{aligned} \Rightarrow \begin{aligned} -.2x + .1y &= 0 \\ .2x - .1y &= 0 \end{aligned}$$

$\Downarrow$

Implicit $x + y = 1$

$(.8x + .1y \ \ \ .2x + .9y) = (x, y)$

So $x + y = 1$

$+\,5(-.2x + .1y = 0)$

$1.5y = 1 \Rightarrow y = 2/3,\ x = 1/3$

$(1/2 \ \ 2/3)$